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Higher-order expansions of powered extremes of normal samples

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ABSTRACT

In this paper, higher-order expansions for distributions and densities of powered extremes of standard normal random sequences are established under an optimal choice of normalized constants. Our findings refine the related results in Hall (1980). Furthermore, it is shown that the rate of convergence of distributions/densities of normalized extremes depends in principle on the power index.

1. Introduction

Let $X_1, X_2, \ldots$ be independent random variables with common standard normal distribution function (df) $\Phi(x)$, and let $M_n = \max(X_1, X_2, \ldots, X_n)$. It is well-known that $\Phi(x)$ belongs to the max-domain of attraction of Gumbel extreme value distribution $\Lambda(x) = \exp\left(-e^{-x}\right)$, i.e., there exist norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that (see e.g., Leadbetter et al., 1983)

$$\lim_{n\to\infty} \mathbb{P}\left(M_n \le a_n x + b_n\right) = \Lambda(x), \quad x \in \mathbb{R}. \tag{1.1}$$

Hall (1979) showed that $1/\log n$ is the best convergence rate of (1.1). Further, motivated by Haldance and Jayakar (1963), Hall (1980) established the asymptotic distribution behavior of normalized $|M_n|^t$, the powered extremes for given power index $t > 0$. Precisely speaking, with $b_n > 0$ the solution of the following equation

$$2\pi b_n^2 \exp\left(b_n^2\right) = n^2, \quad \forall n \in \mathbb{N}. \tag{1.2}$$

Hall (1980) showed that

$$\lim_{n\to\infty} b_n^{2+2\mathbb{I}\{t=2\}}\left(\mathbb{P}\left(|M_n|^t \le c_n x + d_n\right) - \Lambda(x)\right) = \Lambda(x)e^{-x}k_1(t,x), \quad x \in \mathbb{R}, \tag{1.3}$$

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where, with $\mathbb{I}\{\cdot\}$ the indicator function,

$$c_n = tb_n^{t-2} - 2b_n^{-2}\mathbb{I}\{t=2\}, \qquad d_n = b_n^t - 2b_n^{-2}\mathbb{I}\{t=2\}, \tag{1.4}$$

$$k_1(t,x) = \left(1 + x + \frac{2-t}{2}x^2\right)\mathbb{I}\{t\neq 2\} - \left(\frac{7}{2} + 3x + x^2\right)\mathbb{I}\{t=2\}. \tag{1.5}$$

Recently, Liao et al. (2013) and Hashorva et al. (2014) studied respectively expansions of asymptotic distributions of normalized extremes for logarithmic skew-normal distributions and bivariate normal triangular arrays. Liu and Liu (2013) and Chen and Huang (2014) studied respectively uniform convergence rates of distributions of normalized extremes for Maxwell samples and asymmetry normal samples. For other related work on extreme value distributions, densities and moments of given distributions and their associated uniform convergence rates, we refer to Nair (1981), Smith (1982), Omey (1988), de Haan and Resnick (1996), Cheng (2003), Withers and Nadarajah (2010) and Li and Li (2015), and references therein.

In this paper, we aim to establish higher-order expansions of distributions and densities of powered extremes for normal random samples. The motivation is two-folded. One comes from the importance of determining the efficiency of distribution/density approximations to its Gumbel limit law as shown by the contributions mentioned above. The other is that the powered normal laws are one challenging alternative to normal ones due to its mathematical properties including skewness, heavy tails, etc.

The contribution of this paper is to establish the rate of convergence of (1.3). Moreover, we find out that higher-order expansions of densities of powered extremes display similar asymptotic structures as those for their distributions.

The rest of the paper is organized as follows. Section 2 presents main results. All the proofs are relegated to Section 3.

2. Main results

In this section, we shall establish higher-order expansions of distributions and densities of powered extremes under normalization (see Theorems 2.1 and 2.2). It shows that the convergence rates are different between the two cases that the power index $t=2$ and $0<t\neq 2$.

In the sequel, we shall keep the notation given in Section 1 unless stated otherwise. Further, let for $x\in\mathbb{R}$ and $t>0$

$$k_2(t,x) = \begin{cases} \dfrac{43}{3} + 14x + 6x^2 + \dfrac{4}{3}x^3, & t=2;\\ -\left(3 + 3x + \dfrac{3}{2}x^2 + \dfrac{(2-t)(2t+1)}{6}x^3 + \dfrac{(t-2)^2}{8}x^4 - \dfrac{e^{-x}}{2}\left(1 + x + \dfrac{2-t}{2}x^2\right)^2\right), & t\neq 2 \end{cases} \tag{2.1}$$

and

$$\varpi(t,x) = \begin{cases} \dfrac{1}{2} + x + x^2 - e^{-x}\left(\dfrac{7}{2} + 3x + x^2\right), & t=2;\\ x\left(1 - t + \dfrac{t-2}{2}x\right) + e^{-x}\left(1 + x + \dfrac{2-t}{2}x^2\right), & t\neq 2 \end{cases} \tag{2.2}$$

$$\begin{aligned} \tau(t,x) \;=\; & \mathbb{I}\{t=2\}\left(e^{-x}\left(\frac{43}{3} + 14x + 6x^2 + \frac{4}{3}x^3\right) - \left(\frac{1}{3} + 2x + 2x^2 + \frac{4}{3}x^3\right)\right)\\ & + \mathbb{I}\{t\neq 2\}\Bigg(xe^{-x}\left(1 - t + \frac{t-2}{2}x\right)\left(1 + x + \frac{2-t}{2}x^2\right)\\ & + x^2\left(\frac{(1-t)(1-2t)}{2} + \frac{5(1-t)(t-2)}{6}x + \frac{(t-2)^2}{8}x^2\right) - e^{-x}\Bigg(3 + 3x\\ & + \frac{3}{2}x^2 + \frac{(2-t)(2t+1)}{6}x^3 + \frac{(t-2)^2}{8}x^4 - \frac{e^{-x}}{2}\left(1 + x + \frac{2-t}{2}x^2\right)^2\Bigg)\Bigg). \end{aligned} \tag{2.3}$$

Theorem 2.1. *Let $M_n = \max(X_1,\ldots,X_n)$ with $\{X_n, n\geq 1\}$ a sequence of independent random variables with common df $\Phi(x)$. Let further b_n, c_n and d_n are those given by (1.2) and (1.4), respectively. We have for any $t>0$ and as $n\to\infty$*

$$\mathbb{P}\left(|M_n|^t \leq c_n x + d_n\right) = \Lambda(x) + b_n^{-2-2\mathbb{I}\{t=2\}}\Lambda'(x)\left(k_1(t,x) + b_n^{-2}k_2(t,x) + O(b_n^{-4})\right), \quad x\in\mathbb{R},$$

where $\Lambda'(x) = \Lambda(x)e^{-x}$, and $k_1(t,x)$ and $k_2(t,x)$ are those given by (1.5) and (2.1), respectively.

Remark 2.1. Theorem 2.1 gives the accurate convergence rate of (1.3), which is proportional to $1/\log n$ for all $t>0$ since $b_n^2 \sim 2\log n$ as $n\to\infty$ due to (1.2).

Next, we shall establish the rate of convergence of density approximation of $(|M_n|^t - d_n)/c_n$ to Gumbel extreme value density function.

Theorem 2.2. *Under the assumptions of Theorem 2.1, we have as $n \to \infty$*

$$\frac{d}{dx}\mathbb{P}\left(|M_n|^t \le c_n x + d_n\right) = \Lambda'(x)\left(1 + b_n^{-2-2\mathbb{I}\{t=2\}}\left(\varpi(t,x) + b_n^{-2}\tau(t,x) + O(b_n^{-4})\right)\right), \quad x \in \mathbb{R},$$

where $\varpi(t,x)$ and $\tau(t,x)$ are those given by (2.2) and (2.3), respectively.

Remark 2.2. (1) We see that higher-order approximations of densities of $(|M_n|^t - d_n)/c_n$ to Gumbel extreme value density function possess the same structure as those for their distributions, i.e., the second-order convergence rate is rather faster for $t = 2$ than that for the other cases, while the third-order one is the same $1/\log n$ for all $t > 0$.

(2) It might be possible to investigate the expansions under consideration for powered kth extremes of normal distributions following the similar arguments, see Theorem 1 by Hall (1980).

3. Proofs

In this section, we shall present the proofs of Theorems 2.1 and 2.2. To this end, we first establish Lemmas 3.1 and 3.2 specifying the expansions of the two terms of densities of $(|M_n|^t - d_n)/c_n$ (see (3.10) below). Hereafter, all the limit relations are for $n \to \infty$ unless otherwise stated.

Lemma 3.1. *Let $\nu_n(x,t) = \Phi^{n-1}\left((c_n x + d_n)^{1/t}\right) - \left(1 - \Phi\left((c_n x + d_n)^{1/t}\right)\right)^{n-1}$, $x \in \mathbb{R}$, $t > 0$ with c_n and d_n given by (1.4).*

(1) *For $0 < t \ne 2$, we have as $n \to \infty$*

$$\begin{aligned}\nu_n(x,t) = \Lambda(x)\Bigg(1 &+ \frac{e^{-x}}{b_n^2}\left(1 + x + \frac{2-t}{2}x^2\right) - \frac{e^{-x}}{b_n^4}\Bigg(3 + 3x + \frac{3}{2}x^2 \\ &+ \frac{(2-t)(2t+1)}{6}x^3 + \frac{(t-2)^2}{8}x^4 - \frac{e^{-x}}{2}\left(1 + x + \frac{2-t}{2}x^2\right)^2\Bigg) + O(b_n^{-6})\Bigg).\end{aligned} \tag{3.1}$$

(2) *For $t = 2$, we have as $n \to \infty$*

$$\nu_n(x,t) = \Lambda(x)\left(1 - \frac{e^{-x}}{b_n^4}\left(\frac{7}{2} + 3x + x^2\right) + \frac{e^{-x}}{b_n^6}\left(\frac{43}{3} + 14x + 6x^2 + \frac{4}{3}x^3\right) + O(b_n^{-8})\right). \tag{3.2}$$

Proof. Note for fixed $x \in \mathbb{R}$ and large n that $c_n x + d_n > 0$. Set below $g_n(t) = (c_n x + d_n)^{1/t}$.

(1) *For $0 < t \ne 2$.* Using the following Taylor's expansion

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3(1+O(x)), \quad x \to 0, \ \alpha \in \mathbb{R}$$

and the fact by (1.2) that $b_n^2 \sim 2\log n$ for large n, we have by (1.4)

$$g_n^\alpha(t) = b_n^\alpha\left(1 + \frac{\alpha x}{b_n^2} + \frac{\alpha(\alpha-t)}{2b_n^4}x^2 + \frac{\alpha(\alpha-t)(\alpha-2t)}{6b_n^6}x^3(1+O(b_n^{-2}))\right). \tag{3.3}$$

Applying (3.3) with $\alpha = -1$ and $\alpha = 2$, we have

$$\begin{aligned}\frac{\phi(g_n(t))}{g_n(t)} &= b_n^{-1}\left(1 - \frac{x}{b_n^2} + \frac{1+t}{2b_n^4}x^2 + O(b_n^{-6})\right) \\ &\quad\times \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{b_n^2}{2}\left(1 + \frac{2}{b_n^2}x + \frac{2-t}{b_n^4}x^2 + \frac{2(t-1)(t-2)}{3b_n^6}x^3 + O(b_n^{-8})\right)\right) \\ &= \frac{e^{-x}}{n}\left(1 + \frac{x}{b_n^2}\left(\frac{t-2}{2}x - 1\right) + \frac{x^2}{b_n^4}\left(\frac{t+1}{2} - \frac{(t-2)(2t+1)}{6}x + \frac{(t-2)^2}{8}x^2\right) + O(b_n^{-6})\right),\end{aligned} \tag{3.4}$$

where the last step follows by (1.2) and $e^x = 1 + x + x^2/2(1+O(x))$, $x \to 0$. Further, it follows by (3.3) with $\alpha = -2$ and $\alpha = -4$ that

$$1 - g_n^{-2}(t) + 3g_n^{-4}(t) + O(g_n^{-6}(t)) = 1 - b_n^{-2} + b_n^{-4}(2x+3) + O(b_n^{-6}).$$

Therefore, it follows further by Lemma 3.1 in Zhou and Ling (2015) with $L = 2$ that

$$\begin{aligned} 1-\Phi\left(g_n(t)\right) &= \frac{\phi(g_n(t))}{g_n(t)}\left(1-g_n^{-2}(t)+3g_n^{-4}(t)+O(g_n^{-6}(t))\right) \\ &= \frac{e^{-x}}{n}\left(1-\frac{1}{b_n^2}\left(1+x+\frac{2-t}{2}x^2\right)\right. \\ &\quad \left.+\frac{1}{b_n^4}\left(3+3x+\frac{3}{2}x^2+\frac{(2-t)(2t+1)}{6}x^3+\frac{(t-2)^2}{8}x^4\right)+O(b_n^{-6})\right) \\ &=: n^{-1}e^{-x}\left(1-\vartheta_1 b_n^{-2}+\vartheta_2 b_n^{-4}+O(b_n^{-6})\right), \end{aligned} \tag{3.5}$$

which together with the fact that $b_n^2 \sim 2\log n$ and $\log(1-x) = -x(1+O(x)),\ x \to 0$ implies that

$$\begin{aligned} \Phi^{n-1}\left(g_n(t)\right) &= \exp\big((n-1)\log\left(1-\left(1-\Phi\left(g_n(t)\right)\right)\right)\big) \\ &= \exp\left(-e^{-x}\left(1-\vartheta_1 b_n^{-2}+\vartheta_2 b_n^{-4}+O(b_n^{-6})\right)\right) \\ &= \Lambda(x)\left(1+\frac{\vartheta_1 e^{-x}}{b_n^2}+\frac{e^{-x}}{b_n^4}\left(\frac{\vartheta_1^2}{2}e^{-x}-\vartheta_2\right)+O(b_n^{-6})\right) \end{aligned} \tag{3.6}$$

and

$$\left(1-\Phi\left(g_n(t)\right)\right)^{n-1} = \left(n^{-1}e^{-x}(1+O(b_n^{-2}))\right)^{n-1} = o(b_n^{-\alpha}), \quad \alpha \ge 6. \tag{3.7}$$

Hence (3.1) follows by recalling ϑ_1 and ϑ_2 given by (3.5).

(2) *For* $t = 2$. We shall verify (3.2) by similar arguments to those for the case $0 < t \neq 2$. Recall by (1.4) that $c_n = 2(1-b_n^{-2})$, $d_n = b_n^2 - 2b_n^{-2}$ for $t = 2$. Hence, for $\alpha \in \mathbb{R}$

$$g_n^{\alpha}(2) = b_n^{\alpha}\left(1+\frac{\alpha x}{b_n^2}-\frac{\alpha}{b_n^4}\left(1+x+\frac{2-\alpha}{2}x^2\right)+\frac{\alpha(2-\alpha)x}{b_n^6}\left(1+x+\frac{4-\alpha}{6}x^2\right)+O(b_n^{-8})\right).$$

Using $e^x = 1+x+x^2/2+x^3/6+O(x^4),\ x \to 0$ and the above equality with $\alpha = -1$, we have

$$\begin{aligned} \frac{\phi(g_n(2))}{g_n(2)} &= \frac{\phi(b_n)}{b_n}e^{-x}\left(1+\frac{1+x}{b_n^2}+\frac{(1+x)^2}{2b_n^4}+\frac{(1+x)^3}{6b_n^6}+O(b_n^{-8})\right) \\ &\quad \times\left(1-\frac{x}{b_n^2}+\frac{1}{b_n^4}\left(1+x+\frac{3}{2}x^2\right)-\frac{3x}{b_n^6}\left(1+x+\frac{5}{6}x^2\right)+O(b_n^{-8})\right) \\ &= \frac{e^{-x}}{n}\left(1+\frac{1}{b_n^2}+\frac{1}{b_n^4}\left(x^2+x+\frac{3}{2}\right)-\frac{1}{b_n^6}\left(\frac{4}{3}x^3+x^2+x-\frac{7}{6}\right)+O(b_n^{-8})\right). \end{aligned} \tag{3.8}$$

Additionally,

$$1-\frac{1}{g_n^2(2)}+\frac{3}{g_n^4(2)}-\frac{15}{g_n^4(2)}+O(b_n^{-8}) = 1-\frac{1}{b_n^2}+\frac{2x+3}{b_n^4}-\frac{17+14x+4x^2}{b_n^6}+O(b_n^{-8}).$$

Consequently, a straightforward application of Lemma 3.1 with $L = 3$ in Zhou and Ling (2015) yields that

$$1-\Phi(g_n(2)) = \frac{e^{-x}}{n}\left(1+\frac{1}{b_n^4}\left(\frac{7}{2}+3x+x^2\right)-\frac{1}{b_n^6}\left(\frac{43}{3}+14x+6x^2+\frac{4}{3}x^3\right)+O(b_n^{-8})\right).$$

The rest proof follows by the same arguments of (3.6) and (3.7) for $g_n(2)$. We complete the proof of Lemma 3.1. □

Lemma 3.2. *Let c_n and d_n be given by (1.4). We have as $n \to \infty$*

$$\begin{aligned} & n\frac{d}{dx}\Phi((c_n x+d_n)^{1/t}) \\ &= \begin{cases} e^{-x}\left(1+\dfrac{x}{b_n^2}\left(1-t+\dfrac{t-2}{2}x\right)\right. \\ \quad \left.+\dfrac{x^2}{b_n^4}\left(\dfrac{(1-t)(1-2t)}{2}+\dfrac{5(1-t)(t-2)}{6}x+\dfrac{(t-2)^2}{8}x^2\right)+O(b_n^{-6})\right), & t \neq 2; \\ e^{-x}\left(1+b_n^{-4}\left(\dfrac{1}{2}+x+x^2\right)-b_n^{-6}\left(\dfrac{1}{3}+2x+2x^2+\dfrac{4}{3}x^3\right)+O(b_n^{-8})\right), & t = 2. \end{cases} \end{aligned}$$

Proof. Clearly, we have

$$n\frac{d}{dx}\Phi((c_n x+d_n)^{1/t}) = t^{-1}nc_n\left(c_n x+d_n\right)^{1/t-1}\phi\left((c_n x+d_n)^{1/t}\right).$$

For $0 < t \neq 2$, we have by (1.4) that $c_n = tb_n^{t-2}$. It follows by (3.3) with $\alpha = 1 - t$, and (3.4) for the expansion of $\phi(g_n(t))$ with $g_n(t) = (c_n x + d_n)^{1/t}$ that

$$\begin{aligned} n\frac{d}{dx}\Phi(g_n(t)) &= e^{-x}\left(1 + \frac{(1-t)x}{b_n^2} + \frac{(1-t)(1-2t)}{2b_n^4}x^2 + O(b_n^{-6})\right) \\ &\quad \times \left(1 + \frac{t-2}{2b_n^2}x^2 - \frac{(t-1)(t-2)}{3b_n^4}x^3 + \frac{(t-2)^2}{8b_n^4}x^4 + O(b_n^{-6})\right) \\ &= e^{-x}\left(1 + \frac{x}{b_n^2}\left(1 - t + \frac{t-2}{2}x\right)\right. \\ &\quad \left. + \frac{x^2}{b_n^4}\left(\frac{(1-t)(1-2t)}{2} + \frac{5(1-t)(t-2)}{6}x + \frac{(t-2)^2}{8}x^2\right) + O(b_n^{-6})\right). \end{aligned}$$

For $t = 2$, recalling that $c_n = 2(1 - b_n^{-2})$, the claim follows by (3.8). The proof is complete. □

Proof of Theorem 2.1. Note that for fixed $x \in \mathbb{R}$ and large n, $c_n x + d_n > 0$ and the distribution function of $(|M_n|^t - d_n)/c_n$ is as follows.

$$\mathbb{P}\left(|M_n|^t \leq c_n x + d_n\right) = \Phi^n\left((c_n x + d_n)^{1/t}\right) - \left(1 - \Phi\left((c_n x + d_n)^{1/t}\right)\right)^n. \tag{3.9}$$

For $0 < t \neq 2$, it follows by (3.5) and similar arguments as for (3.6) and (3.7) that

$$\Phi^n\left(g_n(t)\right) = \Lambda(x)\left(1 + \frac{\vartheta_1 e^{-x}}{b_n^2} + \frac{e^{-x}}{b_n^4}\left(\frac{\vartheta_1^2}{2}e^{-x} - \vartheta_2\right) + O(b_n^{-6})\right),$$

where ϑ_1 and ϑ_2 are given by (3.5), and

$$(1 - \Phi\left(g_n(t)\right))^n = o(b_n^{-\alpha}), \quad \alpha \geq 6.$$

Therefore, (3.1) holds with $\nu_n(x, t)$ replaced by the distribution of $(|M_n|^t - d_n)/c_n$ (see (3.9)).

Similarly, for the case $t = 2$, we have (3.2) holds with $\nu_n(x, t)$ replaced by the distribution of $(|M_n|^t - d_n)/c_n$. Consequently, the desired results in Theorem 2.1 are obtained for all $t > 0$. □

Proof of Theorem 2.2. Recalling that $\Phi(-x) = 1 - \Phi(x)$, we have

$$\frac{d}{dx}\mathbb{P}\left(|M_n|^t \leq c_n x + d_n\right) = n\Phi'((c_n x + d_n)^{1/t})\left(\Phi^{n-1}\left((c_n x + d_n)^{1/t}\right) - \left(1 - \Phi\left((c_n x + d_n)^{1/t}\right)\right)^{n-1}\right). \tag{3.10}$$

Hence, for $0 < t \neq 2$, it follows by Lemmas 3.1 and 3.2 that

$$\begin{aligned} &\frac{1}{\Lambda'(x)}\frac{d}{dx}\mathbb{P}\left(|M_n|^t \leq c_n x + d_n\right) - 1 \\ &= \left(1 + \frac{x}{b_n^2}\left(1 - t + \frac{t-2}{2}x\right) + \frac{x^2}{b_n^4}\left(\frac{(1-t)(1-2t)}{2} + \frac{5(1-t)(t-2)}{6}x + \frac{(t-2)^2}{8}x^2\right) + O(b_n^{-6})\right) \\ &\quad \times \left(1 + \frac{e^{-x}}{b_n^2}\left(1 + x + \frac{2-t}{2}x^2\right) - \frac{e^{-x}}{b_n^4}\left(3 + 3x + \frac{3}{2}x^2\right.\right. \\ &\quad \left.\left. + \frac{(2-t)(2t+1)}{6}x^3 + \frac{(t-2)^2}{8}x^4 - \frac{e^{-x}}{2}\left(1 + x + \frac{2-t}{2}x^2\right)^2\right) + O(b_n^{-6})\right) - 1 \\ &= \frac{1}{b_n^2}\left(x\left(1 - t + \frac{t-2}{2}x\right) + e^{-x}\left(1 + x + \frac{2-t}{2}x^2\right)\right) + \frac{1}{b_n^4}\left(xe^{-x}\left(1 - t + \frac{t-2}{2}x\right)\left(1 + x + \frac{2-t}{2}x^2\right)\right. \\ &\quad + x^2\left(\frac{(1-t)(1-2t)}{2} + \frac{5(1-t)(t-2)}{6}x + \frac{(t-2)^2}{8}x^2\right) - e^{-x} \\ &\quad \left. \times \left(3 + 3x + \frac{3}{2}x^2 + \frac{(2-t)(2t+1)}{6}x^3 + \frac{(t-2)^2}{8}x^4 - \frac{e^{-x}}{2}\left(1 + x + \frac{2-t}{2}x^2\right)^2\right)\right) + O(b_n^{-6}) \\ &= b_n^{-2}\varpi(t, x) + b_n^{-4}\tau(t, x) + O(b_n^{-6}), \end{aligned}$$

deriving the claim for $t \neq 2$. Here $\varpi(t, x)$ and $\tau(t, x)$ are given by (2.2) and (2.3), respectively.

Similarly, for $t = 2$, the claim follows by Lemmas 3.1 and 3.2 together with (3.10). We complete the proof of Theorem 2.2. □

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